

# Path Integral for Relativistic Dionium System

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## Abstract

The path integral for relativistic spinless dionium atom is solved, and the energy spectra are extracted from the resulting amplitude.

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# 1 INTRODUCTION

Although the idea of magnetic monopoles probably was discussed in classic physics early in the history of electricity and magnetism, modern discussions of this concept date back only to 1931 by Dirac [1]. He pointed out that magnetic monopoles in quantum mechanics exhibit some extra and subtle features. In particular, with the existence of a magnetic monopole of strength  $g$ , electric charges and magnetic charges must necessarily be quantized, in quantum mechanics [1, 2]. Nowadays, we know that the quantization condition  $2eg/\hbar c = \text{integer}$  exists even at the classical level. This is explained by using the extra monopole gauge invariance [3]. This invariance expresses the physical irrelevance of the shape of the Dirac strings attached to the monopoles. Employing the invariance, the quantization condition can be proved for any fixed particle orbits, i.e. without invoking fluctuating orbits which would correspond to the standard derivation using Schrödinger wave functions.

In modern elementary particle and cosmology theories, magnetic monopole plays an important role. All the grand unified theories predict the existence of monopoles, so do all of the cosmology theories [4]. Therefore the problems relating to the monopoles are always interesting to the physicists. In this paper, we calculate the path integral (PI) of the relativistic dionium atom, i.e., a system of two relativistic particles with both electric and magnetic charges  $(e_1, g_1)$  and  $(e_2, g_2)$ .

In the past 15 years, considerable progress has been made in solving path integral of potential problems [5, 6]. However, only a few relativistic problems have been discussed using PI [7, 8, 9, 10, 11, 12, 13, 14]. In this paper, we solve, following Kleinert's method [15], the relativistic spinless 3-dimensional dionium system by path integral. The energy spectra are extracted from the resulting amplitude.

## 2 THE RELATIVISTIC PATH INTEGRAL

Adding a vector potential  $\mathbf{A}(\mathbf{x})$  to Kleinert's relativistic path integral for a particle in a potential  $V(\mathbf{x})$  [7], we find that the expression of the fixed-energy amplitude is [10]

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2Mc} \int_0^\infty dL \int D\rho \Phi[\rho] \int D^D x e^{-A_E/\hbar} \quad (1)$$

with the action

$$A_E = \int_{\tau_a}^{\tau_b} d\tau \left[ \frac{M}{2\rho(\tau)} x'^2(\tau) - i\mathbf{A} \cdot \mathbf{x}'(\tau) - \rho(\tau) \frac{(E - V)^2}{2Mc^2} + \rho(\tau) \frac{Mc^2}{2} \right]. \quad (2)$$

For the dionium system under consideration, the potential is

$$-e^2/r, \quad (3)$$

and the vector potential

$$\mathbf{A}(\mathbf{x}) = \hbar q \frac{(x_1 \hat{\mathbf{x}}_2 - x_2 \hat{\mathbf{x}}_1) x_3}{r \mathbf{x}_\perp^2}, \quad (4)$$

where  $\mathbf{x}_\perp \equiv (x_1, x_2, 0)$ , and  $\hat{\mathbf{x}}_i$  denotes the basis vectors in the Cartesian coordinate frame.

The constants  $q \equiv -(e_1 g_2 - e_2 g_1)/\hbar c$  and  $e^2 \equiv -e_1 e_2 - g_1 g_2$  in Eqs. (3) and (4) are combinations of the electric and magnetic charges of the two particles, and  $r \equiv \sqrt{x_1^2 + x_2^2 + x_3^2}$

is the radial distance, as usual. The hydrogen atom is a special case of the dionium atom

with  $e_1 = -e_2 = e$  and  $q = 0$ . An electron around a pure magnetic monopole has  $e_1 = e, g_2 =$

$g, e_2 = g_1 = 0$ . In the vector potential we have taken the gauge freedom  $\mathbf{A} \rightarrow \mathbf{A}(\mathbf{x}) + \nabla \Lambda(\mathbf{x})$

to enforce the transverse gauge  $\nabla \cdot \mathbf{A}(\mathbf{x}) = 0$ . In addition, we have taken advantage of

the extra monopole gauge invariance [3] which allows us to choose the shape of the Dirac

String that imports the magnetic flux to the monopoles. The field  $\mathbf{A}(\mathbf{x})$  in Eq. (4) has

two strings of equal strength importing the flux, one along the positive  $x_3$ -axis from plus

infinity to the origin, the other along the negative  $x_3$ -axis from minus infinity to the origin.

As a consequence of monopole gauge invariance, the parameter  $q$  has to be an integer or a half-integer number [3], a condition referred to as Dirac's charge quantization.

Before time-slicing the path integral, we have to regularize it via a so-called  $f$ -transformation [5, 8], which exchanges the path parameter  $\tau$  by a new one  $s$ :

$$d\tau = ds f_l(\mathbf{x}_n) f_r(\mathbf{x}_{n-1}), \quad (5)$$

where  $f_l(\mathbf{x})$  and  $f_r(\mathbf{x})$  are invertible functions whose product is positive. The freedom in choosing  $f_{l,r}$  amounts to an invariance under path-dependent-reparametrizations of the path parameter  $\tau$  in the fixed-energy amplitude of Eq. (1). By this transformation, the (D+1)-dimensional relativistic fixed-energy amplitude for an arbitrary time-independent potential turns into [5, 8]

$$G(\mathbf{x}_b, \mathbf{x}_a; E) \approx \frac{i\hbar}{2Mc} \int_0^\infty dS \prod_{n=1}^{N+1} \left[ \int d\rho_n \Phi(\rho_n) \right] \quad (6)$$

$$\times \frac{f_l(\mathbf{x}_a) f_r(\mathbf{x}_b)}{\left( \frac{2\pi\hbar\epsilon_b^s \rho_b f_l(\mathbf{x}_b) f_r(\mathbf{x}_a)}{M} \right)^{D/2}} \prod_{n=1}^N \left[ \int_{-\infty}^\infty \frac{d^D x_n}{\left( \frac{2\pi\hbar\epsilon_n^s \rho_n f(\mathbf{x}_n)}{M} \right)^{D/2}} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\} \quad (7)$$

with the  $s$ -sliced action

$$A^N = \sum_{n=1}^{N+1} \left[ \frac{M (\mathbf{x}_n - \mathbf{x}_{n-1})^2}{2\epsilon_n^s \rho_n f_l(\mathbf{x}_n) f_r(\mathbf{x}_{n-1})} - i\mathbf{A} \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) - \epsilon_n^s \rho_n f_l(\mathbf{x}_n) f_r(\mathbf{x}_{n-1}) \frac{(E - V)^2}{2Mc^2} + \epsilon_n^s \rho_n f_l(\mathbf{x}_n) f_r(\mathbf{x}_{n-1}) \frac{Mc^2}{2} \right]. \quad (8)$$

A family of functions which regulates the dionium system is

$$f_l(\mathbf{x}) = f(\mathbf{x})^{1-\lambda}, \quad f_r(\mathbf{x}) = f(\mathbf{x})^\lambda, \quad (9)$$

whose product satisfies  $f_l(\mathbf{x}) f_r(\mathbf{x}) = f(\mathbf{x}) = r$ . Thus we obtain the amplitude

$$G(\mathbf{x}_b, \mathbf{x}_a; E) \approx \frac{i\hbar}{2Mc} \int_0^\infty dS \prod_{n=1}^{N+1} \left[ \int d\rho_n \Phi(\rho_n) \right]$$

$$\times \frac{r_a^{1-\lambda} r_b^\lambda}{\left(\frac{2\pi\hbar\epsilon_b^s \rho_b r_b^{1-\lambda} r_a^\lambda}{M}\right)^{3/2}} \prod_{n=2}^{N+1} \left[ \int_{-\infty}^{\infty} \frac{d^3 \Delta x_n}{\left(\frac{2\pi\hbar\epsilon_n^s \rho_n r_{n-1}}{M}\right)^{3/2}} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\}, \quad (10)$$

where the action is

$$A^N = \sum_{n=1}^{N+1} \left[ \frac{M (\mathbf{x}_n - \mathbf{x}_{n-1})^2}{2\epsilon_n^s \rho_n r_n^{1-\lambda} r_{n-1}^\lambda} - i \mathbf{A}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) - \epsilon_n^s \rho_n r_n (r_{n-1}/r_n)^\lambda \frac{(E - V)^2}{2Mc^2} + \epsilon_n^s \rho_n r_n (r_{n-1}/r_n)^\lambda \frac{Mc^2}{2} \right]. \quad (11)$$

In order to use the Kustaanheimo-Stiefel (KS) transformation (e.g. [5]), we now incorporate the dummy fourth dimension into the action by replacing  $\mathbf{x}$  in the kinetic term by the four-vector  $\vec{x}$  and extending the kinetic action to

$$A_{\text{kin}}^N = \sum_{n=1}^{N+1} \frac{M}{2} \frac{(\vec{x}_n - \vec{x}_{n-1})^2}{\epsilon_n^s \rho_n r_n^{1-\lambda} r_{n-1}^\lambda}. \quad (12)$$

This is achieved by satisfaction the following trivial identity

$$\prod_{n=1}^{N+1} \left[ \int \frac{d(\Delta x^4)_n}{\left(2\pi\hbar\epsilon_n^s \rho_n r_n^{1-\lambda} r_{n-1}^\lambda / M\right)^{1/2}} \right] \exp \left\{ -\frac{1}{\hbar} \sum_{n=1}^{N+1} \frac{M}{2} \frac{(\Delta x_n^4)^2}{\epsilon_n^s \rho_n r_n^{1-\lambda} r_{n-1}^\lambda} \right\} = 1. \quad (13)$$

Hence the fixed-energy amplitude of the dionium system in three dimensions can be rewritten as the four-dimensional path integral

$$G(\mathbf{x}_b, \mathbf{x}_a; E) \approx \frac{i\hbar}{2Mc} \int_0^\infty dS \prod_{n=1}^{N+1} \left[ \int d\rho_n \Phi(\rho_n) \right] \times \int d^4 x_a \frac{r_a^{1-\lambda} r_b^\lambda}{\left(\frac{2\pi\hbar\epsilon_b^s \rho_b r_b^{1-\lambda} r_a^\lambda}{M}\right)^2} \prod_{n=2}^{N+1} \left[ \int_{-\infty}^{\infty} \frac{d^4 \Delta x_n}{\left(\frac{2\pi\hbar\epsilon_n^s \rho_n r_{n-1}}{M}\right)^2} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\}, \quad (14)$$

where  $A^N$  is the action of Eq. (11) in which the three-vectors  $\mathbf{x}_n$  are replaced by the four-vectors  $\vec{x}_n$ . With the help of the following approximation

$$\frac{r_a^{1-\lambda} r_b^\lambda}{\left(\frac{2\pi\hbar\epsilon_b^s \rho_b r_b^{1-\lambda} r_a^\lambda}{M}\right)^2} \prod_{n=2}^{N+1} \left[ \int_{-\infty}^{\infty} \frac{d^4 \Delta x_n}{\left(\frac{2\pi\hbar\epsilon_n^s \rho_n r_{n-1}}{M}\right)^2} \right]$$

$$\approx \frac{1}{r_a} \frac{1}{\left(\frac{2\pi\hbar\epsilon_b^s\rho_b}{M}\right)^2} \prod_{n=2}^{N+1} \left[ \int_{-\infty}^{\infty} \frac{d^4\Delta x_n}{\left(\frac{2\pi\hbar\epsilon_n^s\rho_n r_n}{M}\right)^2} \right] \exp \left\{ 3\lambda \sum_{n=1}^{N+1} \log \frac{r_n}{r_{n-1}} \right\}, \quad (15)$$

where the equality  $(r_b/r_a)^{3\lambda-2} = \prod_1^{N+1} (r_n/r_{n-1})^{3\lambda-2}$  has been used, we obtain

$$G(\mathbf{x}_b, \mathbf{x}_a; E) \approx \frac{i\hbar}{2Mc} \int_0^\infty dS \prod_{n=1}^{N+1} \left[ \int d\rho_n \Phi(\rho_n) \right] \\ \times \int \frac{dx_a^4}{r_a} \frac{1}{\left(\frac{2\pi\hbar\epsilon_b^s\rho_b}{M}\right)^2} \prod_{n=2}^{N+1} \left[ \int_{-\infty}^{\infty} \frac{d^4\Delta x_n}{\left(\frac{2\pi\hbar\epsilon_n^s\rho_n r_n}{M}\right)^2} \right] \exp \left\{ -\frac{1}{\hbar} \sum_{n=1}^{N+1} \left[ A^N - 3\lambda\hbar \log \frac{r_n}{r_{n-1}} \right] \right\}. \quad (16)$$

Since the path integral represents the general relativistic resolvent operator, all results must be independent of the splitting parameter  $\lambda$  after going to the continuum limit. Choosing a splitting parameter  $\lambda = 0$ , we obtain the continuum limit of the action

$$A_E[x, x'] = \int_0^S ds \left[ \frac{Mx'^2}{2\rho r} - i\mathbf{A} \cdot \mathbf{x}' - \rho r \frac{(E - V)^2}{2Mc^2} + \rho r \frac{Mc^2}{2} \right]. \quad (17)$$

We now solve the dionium system by introducing the KS transformation (e.g. [5])

$$d\vec{x} = 2A(\vec{u})d\vec{u}. \quad (18)$$

The arrow on top of  $x$  indicates that  $\vec{x}$  is a four-vector. For symmetry reasons, we choose the  $4 \times 4$  matrix  $A(\vec{u})$  as

$$A(\vec{u}) = \begin{pmatrix} u^3 & u^4 & u^1 & u^2 \\ u^4 & -u^3 & -u^2 & u^1 \\ u^1 & u^2 & -u^3 & -u^4 \\ u^2 & -u^1 & u^4 & -u^3 \end{pmatrix}. \quad (19)$$

The transformation of coordinate difference is

$$(\Delta \mathbf{x}_n^i)^2 = 4\bar{\mathbf{u}}_n^2 (\Delta \mathbf{u}_n^i)^2, \quad (20)$$

where  $\bar{\mathbf{u}}_n \equiv (\mathbf{u}_n + \mathbf{u}_{n-1})/2$ . In the continuum limit, this amounts to be

$$d^4x = 16r^2 d^4u, \quad (21)$$

$$\vec{x}'^2 = 4\vec{u}'^2\vec{u}^2 = 4r\vec{u}'^2. \quad (22)$$

By employing the basis tetrad notation  $e^i{}_\mu(\vec{u})$ , Eq. (18) has the form  $dx^i = e^i{}_\mu(\vec{u}) du^\mu$ , and it is given by

$$e^i{}_\mu(\vec{u}) = \frac{\partial x^i}{\partial u^\mu}(\vec{u}) = 2A^i{}_\mu(\vec{u}), \quad i = 1, 2, 3, 4. \quad (23)$$

Under the KS transformation, the magnetic interaction turns into

$$\begin{aligned} \mathbf{A}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) &= \hbar q \frac{[(x_1)_n(\Delta x_2)_n - (x_2)_n(\Delta x_1)_n](x_3)_n}{r_n(\mathbf{x}_\perp)_n^2} \\ &= -\frac{\hbar q}{r_n} \left[ \frac{u_n^1 \Delta u_n^2 - u_n^2 \Delta u_n^1}{(u_n^1)^2 + (u_n^2)^2} + \frac{u_n^4 \Delta u_n^3 - u_n^3 \Delta u_n^4}{(u_n^3)^2 + (u_n^4)^2} \right] \\ &\quad \times \left[ (u_n^1)^2 + (u_n^2)^2 - (u_n^3)^2 - (u_n^4)^2 \right]. \end{aligned} \quad (24)$$

We obtain a path integral equivalent to Eq. (14)

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2Mc} \int_0^\infty dS e^{SEe^2/\hbar Mc^2} G(\vec{u}_b, \vec{u}_a; S), \quad (25)$$

where  $G(\vec{u}_b, \vec{u}_a; S)$  denotes the s-sliced amplitude

$$\prod_{n=1}^{N+1} \left[ \int d\rho_n \Phi(\rho_n) \right] \frac{1}{16} \int \frac{dx_a^4}{r_a} \frac{1}{\left(\frac{2\pi\hbar\epsilon_n^s \rho_b}{m}\right)^2} \prod_{n=1}^N \left[ \int_{-\infty}^\infty \frac{d^4 u_n}{\left(\frac{2\pi\hbar\epsilon_n^s \rho_n}{m}\right)^2} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\} \quad (26)$$

with the action

$$A^N = \sum_{n=1}^{N+1} \left\{ \frac{m(\Delta \vec{u}_n)^2}{2\epsilon_n^s \rho_n} - i(\vec{A}_n \cdot \Delta \vec{u}_n) + \epsilon_n^s \rho_n \frac{m\omega^2 \vec{u}_n^2}{2} - \epsilon_n^s \rho_n \frac{\hbar^2 4\alpha^2}{2m\vec{u}_n^2} \right\}. \quad (27)$$

Here

$$m = 4M, \quad \omega^2 = \frac{M^2 c^4 - E^2}{4M^2 c^2}, \quad (28)$$

and  $\vec{A}_n \cdot \Delta \vec{u}_n$  is given by Eq. (24). After taking the continuum limit, this leads to the

Duru-Kleinert transformed action

$$A = \int_0^S ds \left[ \frac{m\vec{u}'^2}{2\rho(s)} - i(\vec{A} \cdot \vec{u}') + \rho(s) \frac{m\omega^2 \vec{u}^2}{2} - \rho(s) \frac{4\hbar^2 \alpha^2}{2m\vec{u}^2} \right]. \quad (29)$$

Let us now analyze the effect coming from the magnetic interaction upon the Coulomb system. We first express  $(u^1, u^2, u^3, u^4)$  in terms of three-dimensional spherical coordinate with an auxiliary angle  $\gamma$ :

$$\left. \begin{aligned} u^1 &= \sqrt{r} \cos(\theta/2) \cos[(\varphi + \gamma)/2] \\ u^2 &= \sqrt{r} \cos(\theta/2) \sin[(\varphi + \gamma)/2] \\ u^3 &= \sqrt{r} \sin(\theta/2) \cos[(\varphi - \gamma)/2] \\ u^4 &= \sqrt{r} \sin(\theta/2) \sin[(\varphi - \gamma)/2] \end{aligned} \right\} \quad \left( \begin{aligned} 0 &\leq \theta \leq \pi \\ 0 &\leq \varphi \leq 2\pi \\ 0 &\leq \gamma \leq 4\pi \end{aligned} \right). \quad (30)$$

This gives us the equivalent form of Eq. (26)

$$\prod_{n=1}^{N+1} \left[ \int d\rho_n \Phi(\rho_n) \right] \frac{1}{16} \int_0^{4\pi} d\gamma_a \frac{1}{\left( \frac{2\pi\hbar\epsilon_b^s \rho_b}{m} \right)^2} \prod_{n=1}^N \left[ \int_{-\infty}^{\infty} \frac{d^4 u_n}{\left( \frac{2\pi\hbar\epsilon_n^s \rho_n}{m} \right)^2} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\} \quad (31)$$

with the continuum action

$$\begin{aligned} A = \int_0^S ds \left\{ \frac{m}{2\rho(s)} \left[ u'^2 + \frac{u^2}{4} \left( \theta'^2 + \varphi'^2 + \gamma'^2 + 2\varphi' \left( \gamma' - \rho \frac{4\hbar qi}{mu^2} \right) \cos \theta \right) \right] \right. \\ \left. + \rho(s) \frac{m\omega^2 \vec{u}^2}{2} - \rho(s) \frac{4\hbar^2 \alpha^2}{2m\vec{u}^2} \right\}. \end{aligned} \quad (32)$$

The spherical coordinate  $(u, \theta, \varphi)$  and the auxiliary angle  $\gamma$  can be represented by canonical momentums as follows:

$$\left\{ \begin{aligned} u' &= \rho \frac{p_u}{m}, \\ \theta' &= \frac{p_\theta}{\xi}, \\ \gamma' &= \frac{1}{\xi \sin^2 \theta} [p_\gamma + \eta \xi \cos^2 \theta - p_\varphi \cos \theta], \\ \varphi' &= \frac{1}{\xi \sin^2 \theta} [p_\varphi - \eta \xi \cos \theta - p_\gamma \cos \theta], \end{aligned} \right. \quad (33)$$

where the variables  $\eta \equiv -4\rho\hbar qi/mu^2$  and  $\xi \equiv mu^2/4\rho$ . With the help of Eq. (33), we obtain

the canonical form of the path integral

$$\begin{aligned} \frac{i\hbar}{2Mc} \int_0^\infty dS \, e^{SEe^2/\hbar Mc^2} \prod_{n=1}^{N+1} \left[ \int d\rho_n \Phi(\rho_n) \right] \frac{1}{16} \int_0^{4\pi} d\gamma_a \\ \times \prod_{n=1}^N \left[ \int_{-\infty}^\infty d^4 \vec{u}_n \right] \prod_{n=1}^{N+1} \left[ \int_{-\infty}^\infty \frac{d^4 (p_u)_n}{2\pi\hbar} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\} \end{aligned} \quad (34)$$

with the action of continuum version

$$A = \int_0^S ds \left\{ -i [p_u u' + p_\theta \theta' + p_\varphi \varphi' + (p_\gamma - \hbar q) \gamma'] + H \right\}, \quad (35)$$



where the Hamiltonian is given by

$$H = \frac{\rho}{2m} \left\{ p_u^2 + \frac{4}{\vec{u}^2} \left[ p_\theta^2 + \frac{1}{\sin^2 \theta} (p_\varphi^2 + p_\gamma^2 - 2p_\gamma p_\varphi \cos \theta) \right] \right\} \\ + \frac{4\rho}{2m\vec{u}^2} \left[ -2\hbar q (p_\gamma - \hbar q) + \hbar^2 (\alpha^2 + q^2) \right] + \rho \frac{m\omega^2 \vec{u}^2}{2}. \quad (36)$$

This differs from the pure relativistic Coulomb system [7] in two places:

First, the Hamiltonian has an extra centrifugal barrier proportional to the charge parameter  $q$ :

$$V(r) = \frac{-2\hbar q \rho (p_\gamma - \hbar q)}{2Mr}. \quad (37)$$

Second, the action of Eq. (35) contains an additional term

$$\Delta A = -\hbar q \int_0^S ds \gamma'. \quad (38)$$

Fortunately, this is a pure surface term  $\Delta A = -\hbar q (\gamma_b - \gamma_a)$ . Due to Eq. (38), the modification consists of a simple extra phase factor in the integral over  $\gamma_a$  so that

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2Mc} \int_0^\infty dS e^{SEe^2/\hbar Mc^2} \frac{1}{16} \int_0^{4\pi} d\gamma_a e^{-iq(\gamma_b - \gamma_a)} G(\vec{u}_b, \vec{u}_a; S). \quad (39)$$

Since the integral over  $\gamma_a$  forces the momentum  $p_r$  in the canonical action (35) to take the value  $\hbar q$ . This eliminates the term proportional to  $p_\gamma - \hbar q$  in Eq. (36), therefore the Green's function in  $u$ -space has the form

$$G(\vec{u}_b, \vec{u}_a; S) \approx \prod_{n=1}^{N+1} \left[ \int d\rho_n \Phi(\rho_n) \right] \frac{1}{\left( \frac{2\pi\hbar\epsilon_b^s \rho_b}{m} \right)^2} \prod_{n=1}^N \left[ \int_{-\infty}^\infty \frac{d^4 u_n}{\left( \frac{2\pi\hbar\epsilon_n^s \rho_n}{m} \right)^2} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\} \quad (40)$$

with the action

$$A^N = \sum_{n=1}^{N+1} \left\{ \frac{m(\Delta \vec{u}_n)^2}{2\epsilon_n^s \rho_n} + \epsilon_n^s \rho_n \frac{m\omega^2 \vec{u}_n^2}{2} - \epsilon_n^s \rho_n \frac{\hbar^2 4(\alpha^2 + q^2)}{2m\vec{u}_n^2} \right\}. \quad (41)$$

We now choose the gauge  $\rho(s) = 1$  in Eq. (40). We arrive at

$$G(\vec{u}_b, \vec{u}_a; S) \approx \frac{1}{\left(\frac{2\pi\hbar\epsilon_b^s}{m}\right)^2} \prod_{n=1}^N \left[ \int_{-\infty}^{\infty} \frac{d^4 u_n}{\left(\frac{2\pi\hbar\epsilon_n^s}{m}\right)^2} \right] \exp \left\{ -\frac{1}{\hbar} A^N \right\}, \quad (42)$$

where the action

$$A^N = \sum_{n=1}^{N+1} \left\{ \frac{m(\Delta \vec{u}_n)^2}{2\epsilon_n^s} + \epsilon_n^s \frac{m\omega^2 \vec{u}_n^2}{2} - \epsilon_n^s \frac{\hbar^2 4(\alpha^2 + q^2)}{2m\vec{u}_n^2} \right\}. \quad (43)$$

It describes a particle with mass  $m = 4M$  moving as a function of “pseudotime”  $s$  in a 4-dimensional harmonic oscillator potential of frequency

$$\omega^2 = \frac{M^2 c^4 - E^2}{4M^2 c^2}. \quad (44)$$

The oscillator possesses an additional attractive potential  $-4\hbar^2(\alpha^2 + q^2)/2m\vec{u}^2$  which is conveniently parametrized in the form of a centrifugal barrier

$$V_{\text{extra}} = \hbar^2 \frac{l_{\text{extra}}^2}{2m\vec{u}^2}, \quad (45)$$

whose squared angular momentum has the negative value  $l_{\text{extra}}^2 \equiv -4(\alpha^2 + q^2)$ , where  $\alpha$  denotes the fine-structure constant  $\alpha \equiv e^2/\hbar c$ .

There are no  $\lambda$ -slicing corrections. This is ensured by the affine connection of KS transformation satisfying

$$\Gamma_{\mu}^{\mu\lambda} = g^{\mu\nu} e_i^{\lambda} \partial_{\mu} e^i_{\nu} = 0 \quad (46)$$

and the transverse gauge  $\partial_i A^i = 0$  [5, 8]. The path integral Eq. (42) can be performed and is given by [5]

$$G(\vec{u}_b, \vec{u}_a; S) = \frac{1}{u_b u_a} \sum_{l=0}^{\infty} G(u_b, u_a; S, l) \\ \times \frac{l+1}{2\pi^2} \sum_{k_1, k_2 = -l/2}^{l/2} d_{k_1, k_2}^{l/2}(\theta_b) d_{k_1, k_2}^{l/2}(\theta_a) e^{ik_1(\varphi_b - \varphi_a) + ik_2(\gamma_b - \gamma_a)} \quad (47)$$

with the radial amplitude

$$G(u_b, u_a; S, l) = \frac{m\omega}{\hbar \sinh \omega s} e^{-\frac{m\omega}{2\hbar}(u_b^2 + u_a^2) \coth \omega s} I_{\sqrt{(l+1)^2 - 4(\alpha^2 + q^2)}} \left( \frac{m}{\hbar} \frac{\omega u_b u_a}{\sinh \omega s} \right), \quad (48)$$

where  $d_{k_1, k_2}^{l/2}(\theta) e^{ik_1 \varphi + ik_2 \gamma}$  are the representation functions of the rotation group (e.g.[5]). The integral  $\int_0^{4\pi} d\gamma_a e^{-iq(\gamma_b - \gamma_a)}$  in Eq. (39) now can be easily done. We arrive at the fixed-energy amplitude of the relativistic dionium atom, labeled by the subscript  $D$ ,

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{1}{\sqrt{r_b r_a}} \sum_{l_D} G(r_b, r_a; E_D, l_D) \sum_{k=-l_D}^{l_D} Y_{l_D, k, q}(\theta_b, \varphi_b) Y_{l_D, k, q}^*(\theta_a, \varphi_a), \quad (49)$$

where  $Y_{l_D, k, q}(\theta_b, \varphi_b)$  are the so-called monopole harmonics

$$Y_{l_D, k, q}(\theta, \varphi) = \sqrt{\frac{l+1}{4\pi}} e^{ik\varphi} d_{k, q}^{l_D}(\theta), \quad (50)$$

and  $l_D$  is defined as  $l/2$ . The radial amplitude for the dionium is

$$\begin{aligned} G(r_b, r_a; E_D, l_D) &= \frac{i\hbar}{2Mc} \frac{1}{2} \int_0^\infty dS e^{SE_D e^2 / \hbar M c^2} \\ &\times \frac{m\omega}{\hbar \sinh \omega s} e^{-\frac{m\omega}{2\hbar}(r_b + r_a) \coth \omega s} I_{\sqrt{(2l_D+1)^2 - 4(\alpha^2 + q^2)}} \left( \frac{m\omega \sqrt{r_b r_a}}{\hbar \sinh \omega s} \right). \end{aligned} \quad (51)$$

This integral can be calculated by employing the formula

$$\begin{aligned} &\int_0^\infty dy \frac{e^{2\nu y}}{\sinh y} \exp \left[ -\frac{t}{2} (\zeta_a + \zeta_b) \coth y \right] I_\mu \left( \frac{t\sqrt{\zeta_b \zeta_a}}{\sinh y} \right) \\ &= \frac{\Gamma((1+\mu)/2 - \nu)}{t\sqrt{\zeta_b \zeta_a} \Gamma(\mu+1)} W_{\nu, \mu/2}(t\zeta_b) M_{\nu, \mu/2}(t\zeta_b), \end{aligned} \quad (52)$$

with the range of validity

$$\begin{aligned} &\zeta_b > \zeta_a > 0, \\ &\text{Re}[(1+\mu)/2 - \nu] > 0, \\ &\text{Re}(t) > 0, |\arg t| < \pi, \end{aligned}$$

where  $M_{\mu, \nu}$  and  $W_{\mu, \nu}$  are the Whittaker functions, we complete the integration of Eq. (51),

and find the amplitude for  $u_b > u_a$  in the closed form,

$$\begin{aligned}
G(r_b, r_a; E_D, l_D) &= \frac{i\hbar}{2Mc} \frac{1}{2\omega} \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}\sqrt{(2l_D+1)^2 - 4(\alpha^2 + q^2)} - E_D\alpha/\sqrt{M^2c^4 - E_D^2}\right)}{\sqrt{r_b r_a} \Gamma\left(\sqrt{(2l_D+1)^2 - 4(\alpha^2 + q^2)} + 1\right)} \\
&\times W_{E_D\alpha/\sqrt{M^2c^4 - E_D^2}, \sqrt{(2l_D+1)^2 - 4(\alpha^2 + q^2)}/2} \left(\frac{2}{\hbar c} \sqrt{M^2c^4 - E_D^2} r_b\right) \\
&\times M_{E_D\alpha/\sqrt{M^2c^4 - E_D^2}, \sqrt{(2l_D+1)^2 - 4(\alpha^2 + q^2)}/2} \left(\frac{2}{\hbar c} \sqrt{M^2c^4 - E_D^2} r_a\right). \quad (53)
\end{aligned}$$

It is easy to check, with  $n_r = n - l_D - 1$ , that the spectra reduced to relativistic Coulomb case, if we take  $q = 0, e_1 = -e_2 = e$  [7].

The energy spectra can be extracted from the poles. They are determined by

$$\frac{1}{2} + \frac{1}{2}\sqrt{(2l_D+1)^2 - 4(\alpha^2 + q^2)} - E_D\alpha/\sqrt{M^2c^4 - E_D^2} = -n_r, \quad n_r = 0, 1, 2, 3, \dots \quad (54)$$

After some mathematical manipulation, we have

$$E_{n_r, l_D, q} = \pm Mc^2 \left[ 1 + \frac{\alpha^2}{\left(\frac{1}{2} + \frac{1}{2}\sqrt{(2l_D+1)^2 - 4(q^2 + \alpha^2)} + n_r\right)^2} \right]^{-1/2}. \quad (55)$$

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